Parametrising clusters of sections

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Motivation:

$$\left.\begin{array}{l} \bullet \text{ Kleiman's iterated blow ups} \\ \qquad\qquad + \\ \dagger \text{ Degenerations of surfaces} \end{array}\right\} \text{Universal scheme of clusters of sections}$$

- Parametrise (families) of clusters of points $\rightsquigarrow$ of sections

† + "all" its infinitesimal information + families of degenerations

Definition (Ordered clusters)

(Ordered) cluster of points of

$$X \longrightarrow \mathbb{k}$$

Sequence of points $(p_1, \dots, p_r)$

$$\begin{tikzcd}[row sep=tiny] X & \mathrm{bl}_{p_1} X \arrow[l] & \dots \arrow[l] \\ \cup & \cup & \\ p_1 & p_2 & \dots \end{tikzcd}$$

Cluster of sections of

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr] & & Y \arrow[dl] \\ & \mathbb{k} & \end{tikzcd}$$

Sequence of morphisms $(\sigma_1, \dots, \sigma_r)$

$$\begin{tikzcd} Y \arrow[r, bend right, "\sigma_1"'] \arrow[rr, bend right, "\sigma_2"'] & X \arrow[l, "\pi"'] & \mathrm{bl}_{\sigma_1} X \arrow[l] & \dots \arrow[l] \\ & & \dots & \end{tikzcd}$$

GOAL: Parametrise clusters of ...

- ... of points $\rightsquigarrow$ Kleiman's iterated blow ups
- ... of sections $\rightsquigarrow$ Universal scheme of clusters of sections

Definition (in family)

The functor $\mathcal{C}_r \colon \mathsf{Sch}_{\Bbbk} \longrightarrow \mathsf{Set}$ sends $T \longrightarrow \Bbbk$ to

$$\mathcal{C}_r(T/\Bbbk) := \{\text{cluster of sections } (\tau_1, \ldots, \tau_r) \text{ of } \pi_T \colon X_T \longrightarrow Y_T\}$$

$$\begin{tikzcd}
\Bbbk & Y \arrow[l] & X \arrow[l, "\pi"'] & & \\
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Definition at the level of morphisms:

Strong regularity conditions on π

Definition ($\mathcal{C}_r$ on morphisms)

$$\mathcal{C}_r(T/\Bbbk) = \{\text{cluster of sections } (\tau_1, \dots, \tau_r) \text{ of } \pi_T : X_T \longrightarrow Y_T\}$$

From $\begin{tikzcd} T' \arrow[rr, "f"] \arrow[dr] & & T \arrow[dl] \\ & \Bbbk & \end{tikzcd}$ and $\tau = (\tau_1, \dots, \tau_r) \in \mathcal{C}_r(T)$ build

$$\mathcal{C}_r(f)(\tau) = (\tau_1', \dots, \tau_r') \in \mathcal{C}_r(T')$$

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\end{tikzcd}$$

Steady family $\pi : X \longrightarrow Y$: regularity conditions to define $\mathcal{C}_r$

First result (existence)

Theorem

If

- $X \xrightarrow{\pi} Y$ *is a steady family (to define $\mathcal{C}_r$)*
- Y *is proper*
- X *is quasiprojective*

then

$$\text{the functor } \mathcal{C}_r \text{ is representable by a scheme } Cl_r$$

the r-th Universal scheme of families of section-clusters (Ucs) over π

Proof.

$\square$

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Proof.

$r = 1 \rightsquigarrow$ Grothendieck's FGA.

$\square$

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Proof.

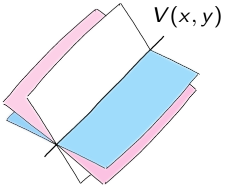
Assume $(Cl_r, (\sigma_1, \dots, \sigma_r))$ exists,

$$\begin{tikzcd} Cl_r & Cl_r \times Y \arrow[l] \arrow[d] \arrow[rr, bend right, "\sigma_1"] \arrow[rrrr, bend right, "\sigma_r"] & X_1 := Cl_r \times X \arrow[l, "\pi_{Cl_r}"'] & \cdots \arrow[l] & X_r \arrow[l] & X_{r+1} := \mathrm{bl}_{\sigma_r}(X_r) \arrow[l] \\ & Y & & \cdots & & \end{tikzcd}$$

$\square$

Example: pencil of planes through a line

$$\begin{aligned} \pi \quad : \quad X = \mathbb{P}^3_{\Bbbk} \setminus V(x,y) &\longrightarrow \mathbb{P}^1_{\Bbbk} \\ (x:y:z:w) &\longmapsto (x:y) \end{aligned}$$



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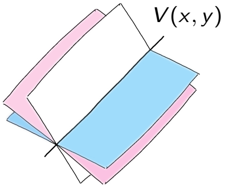

Fibre of π at $(\alpha : \beta) \in \mathbb{P}^1_{\Bbbk}$

$$\pi^{-1}((\alpha:\beta)) = V\left(\begin{vmatrix} x & y \\ \alpha & \beta \end{vmatrix} \right) \cap X$$

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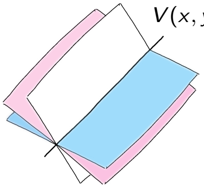

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- Case $r = 1$ (parametrise sections of π):

$$Cl_1 \cong \mathbb{A}^4_{a,b,c,d} \xhookrightarrow{\textbf{open}} \mathbb{G}(1,3), \text{ with}$$

$$(a,b,c,d) \in \mathbb{A}^4_{\Bbbk} \longmapsto \left\{ \begin{array}{l} ax + by - z \\ cx + dy - w \end{array} \right.$$

Example: pencil of planes containing a line

- Case $r = 2$ (pairs of (possibly infinitely near) sections):

$$\mathbb{A}^4_{a,b,c,d} \times \mathbb{A}^4_{a',b',c',d'} \supseteq \Delta$$

Example: pencil of planes containing a line

- Case $r = 2$ (pairs of (possibly infinitely near) sections):

$$\mathbb{A}^4_{a,b,c,d} \times \mathbb{A}^4_{a',b',c',d'} \supseteq \Delta$$

  - $\sigma : \mathbb{P}^1_{\Bbbk} \longrightarrow X$ a section of π, that is, a line $\sigma(\mathbb{P}^1_{\Bbbk}) \subseteq X$
  - $\tau : \mathbb{P}^1_{\Bbbk} \longrightarrow \mathrm{bl}_\sigma X$ a section of $\mathrm{bl}_\sigma X \longrightarrow X \longrightarrow \mathbb{P}^1_{\Bbbk}$,
    (the exceptional divisor is $\mathbb{P}^1 \times \mathbb{P}^1$)

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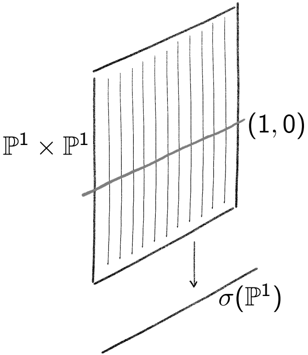

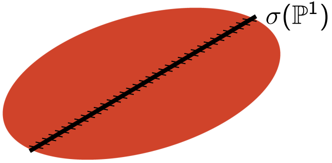

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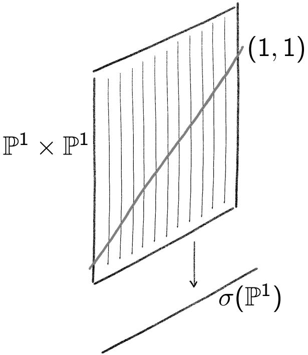

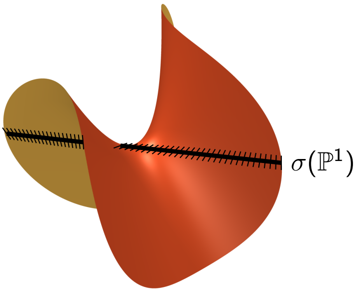

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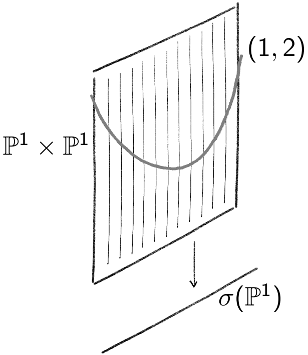

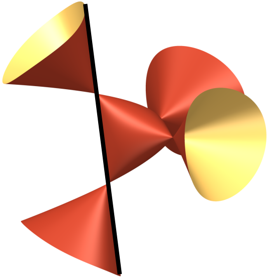

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$$Cl_2 = V_0 \sqcup V_1 \sqcup \bigsqcup_{n \geq 2} V_n$$

$$\mathbb{A}^4 \times \mathbb{A}^4 = \Big(T \setminus \Delta\Big) \sqcup \Big((\mathbb{A}^4 \times \mathbb{A}^4) \setminus T\Big) \sqcup \Big(\Delta\Big)$$

- "all" infinitesimal information... as a Y-scheme!

The blow up split section family

$$\begin{tikzcd}
 & Z \arrow[d, hook, "\textbf{cl.emb.}"'] \\
Y \arrow[d] & \mathfrak{X} \times Y \arrow[l, "\pi"'] \arrow[d, "\alpha"] \\
\mathbb{k} & \mathfrak{X} \arrow[l]
\end{tikzcd}$$

The blow up split section family

$$\begin{tikzcd}
 & Z \arrow[d, hook, "\textbf{cl.emb.}"'] & Z \times_X \mathfrak{B} \arrow[l] \arrow[d, hook, "\textbf{Cartier}"'] \\
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\Bbbk & \mathfrak{X} \arrow[l] & \mathfrak{B} \arrow[l, "b"']
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The couple $(\mathfrak{B}, b)$ is the *blow up §family of* π *along* Z.

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Theorem

If

- $\mathfrak{X}$ *quasiprojective*
- Y *proper, smooth and irreducible,*

then $(\mathfrak{B}, b)$ *exists.*

The blow up §family: structure

Flattening stratification:

$$\begin{tikzcd} Z \arrow[d, "\alpha|_Z"'] & \coprod_P Z_P \arrow[l, hook'] \arrow[d] \\ \mathfrak{X} & \coprod_P \mathfrak{X}_P \arrow[l, hook'] \end{tikzcd}$$

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- Unique stratum $\Delta := \mathfrak{X}_P$, *the core*, such that $Z_P \longrightarrow \mathfrak{X}_P \times Y$ is iso.

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- For every P, there is $U_P \subseteq \mathfrak{X}_P$ such that $Z_P \cap U_P \longrightarrow U_P \times Y$ is Cartier.

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Theorem

If

- $\mathfrak{X}$ *is quasiprojective*
- Y *is projective and smooth*

then,

$$\mathfrak{B} \setminus b^{-1}(\Delta) \cong \bigsqcup_P U_P.$$

The blow up §family and Ucs (iterative construction)

Consider the blow up §family $(\mathfrak{B}, b)$ of

$$\left(Cl_r \times_{Cl_{r-1}} Cl_r\right) \times Y \longrightarrow Cl_r \times_{Cl_{r-1}} Cl_r$$

along

$$Z := \{(\tau, \tau', y) : \tau_r(y) = \tau'_r(y)\}$$

Theorem

If

- $\mathfrak{X}$ *is quasiprojective*
- Y *is projective and smooth*

then,

$$Cl_{r+1} \cong_{\textit{set th.}} \mathfrak{B} \cup \textit{"exceptional components"}.$$

Second result on Ucs (notation)

Consider $F\colon Cl_{r+1} \longrightarrow Cl_r \times_{Cl_{r-1}} Cl_r$ sending

$$(\tau_1, \ldots, \tau_r, \tau_{r+1}) \longmapsto (\tau_1, \ldots, \tau_r;\ \tau_1, \ldots, \mathrm{bl}_{\tau_r} \circ \tau_{r+1})$$

- Observation $F|_{\mathfrak{B}} = b$ (On the example, $(\sigma, \tau) \longrightarrow (\sigma;\ \mathrm{bl}_\sigma \circ \tau)$)

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- Observation $F|_{\mathfrak{B}} = b$ (On the example, $(\sigma, \tau) \longrightarrow (\sigma;\ \mathrm{bl}_\sigma \circ \tau)$)

Consider the flattening stratification of $Z \longrightarrow Cl_r \times_{Cl_{r-1}} Cl_r$

$$\begin{tikzcd}
Z \arrow[r, hook] & \left(Cl_r \times_{Cl_{r-1}} Cl_r\right) \times Y \arrow[r] & Cl_r \times_{Cl_{r-1}} Cl_r \\
Z_{(\tau,\tau')} \arrow[u, hook] \arrow[r, hook] & Y \arrow[u, hook] \arrow[r] & (\tau, \tau') \arrow[u, hook]
\end{tikzcd}$$

The core is Δ_{Cl_r}

$$Cl_r \times_{Cl_{r-1}} Cl_r = \Delta_{Cl_r} \sqcup \bigsqcup_P D_P$$

Second result on Ucs (structure)

Theorem

If

- $X \xrightarrow{\pi} Y$ *is a steady family (to define* $\mathcal{C}_r$*)*
- X *is quasiprojective*
- Y *is projective and smooth*

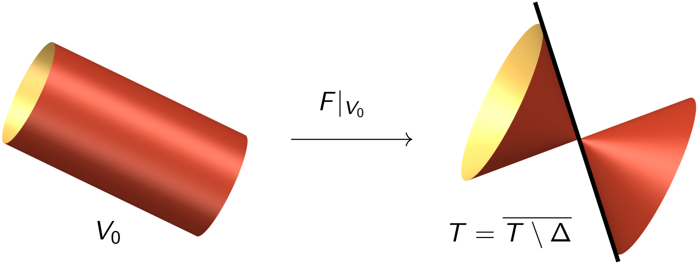
then for every irreducible component $C \subseteq Cl_{r+1}$*, there is* P *with* $F(C) \subseteq \overline{D_P}$

- *if* $D_P \neq \Delta_{Cl_r}$*, then*

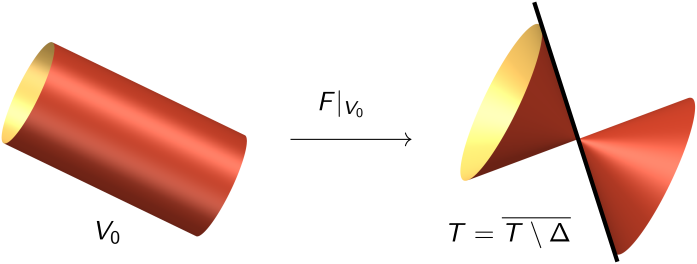
$$\begin{tikzcd} C \arrow[r, hook, "\textbf{open}"] \arrow[rr, bend left, "F|_C"] & \mathrm{bl}_I(\overline{D_P}) \arrow[r] & \overline{D_P} \end{tikzcd}$$

where I *fails to be Cartier only on the diagonal* $\Delta_{Cl_r} \cap \overline{D_P}$*.*

The component of (limits of) meeting lines



The component of (limits of) meeting lines



Moltes gràcies